

MATH-305

Numerical Methods

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3. Bisection Mthd

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Ch. 1 Zeros of Functions

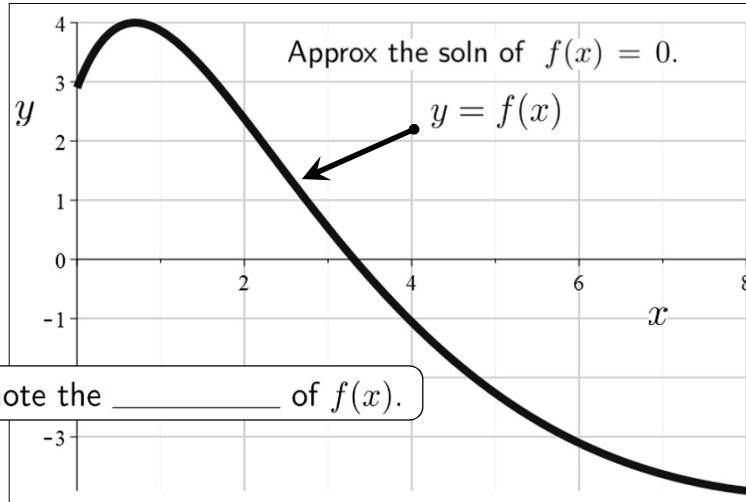
Here we seek a soln of a single _____ eqn involving one unknown.

Example 1: Solve the eqn for x : $e^{-x} = \cos x$.

Zero Form	Root Form
$e^{-x} - \cos x = 0$	$e^{-x} = \cos x$
$f(x) = 0$	Any eqn _____ in zero form.
A soln is called a _____ of f .	A soln is called a _____ of eqn.
A zero of f is an _____ of the _____ of f .	Most of the mthds require the eqn in _____ form.



Zeros of Functions



We'll let z denote the _____ of $f(x)$.

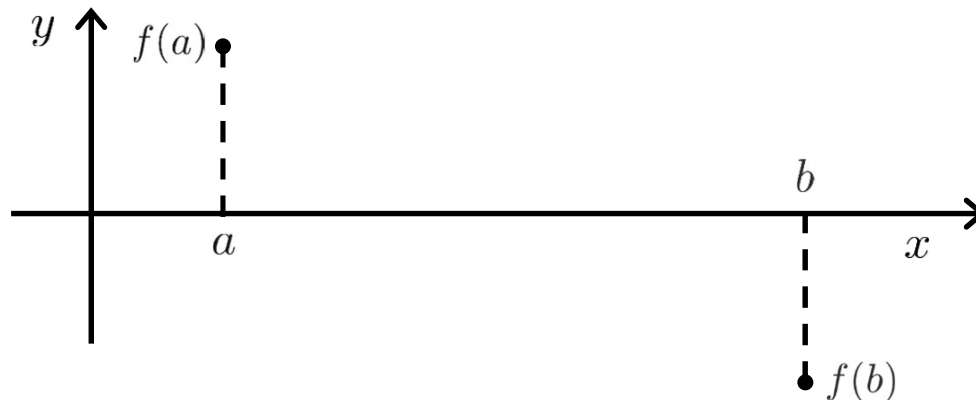
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Intermediate Value Theorem (IVT)

Supp. ftn $f(x)$ is _____ on interval $[a, b]$ and that $f(a)$ and $f(b)$ have _____ signs. Then $f(x)$ has _____ one zero b/w _____.



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Bisection Method | Interval Halving

To approx. a zero of ftn $f(x)$, select a starting interval $[a, b]$ satisfying the _____. We reserve a and b for the _____ of the _____ interval.

For computational convenience, we denote a by x_ℓ and b by x_r , and let

Ensure that y_ℓ and y_r have _____ signs.*

* So the points (x_ℓ, y_ℓ) and (x_r, y_r) are on the _____ of ftn $f(x)$ and are on opposite sides of the ____ axis.

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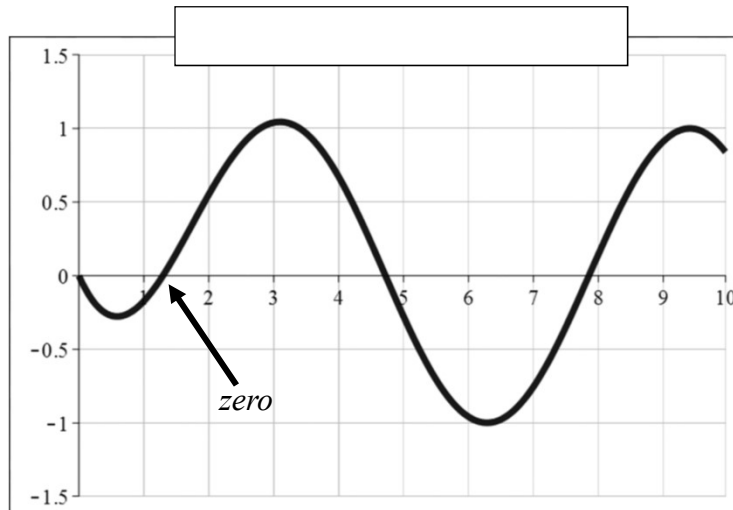
Example 2. Approx. the first positive soln of $e^{-x} = \cos x$.

Solve $f(x) = 0$ for x

First write the eqn in _____ form:

Q. What is a good starting interval?

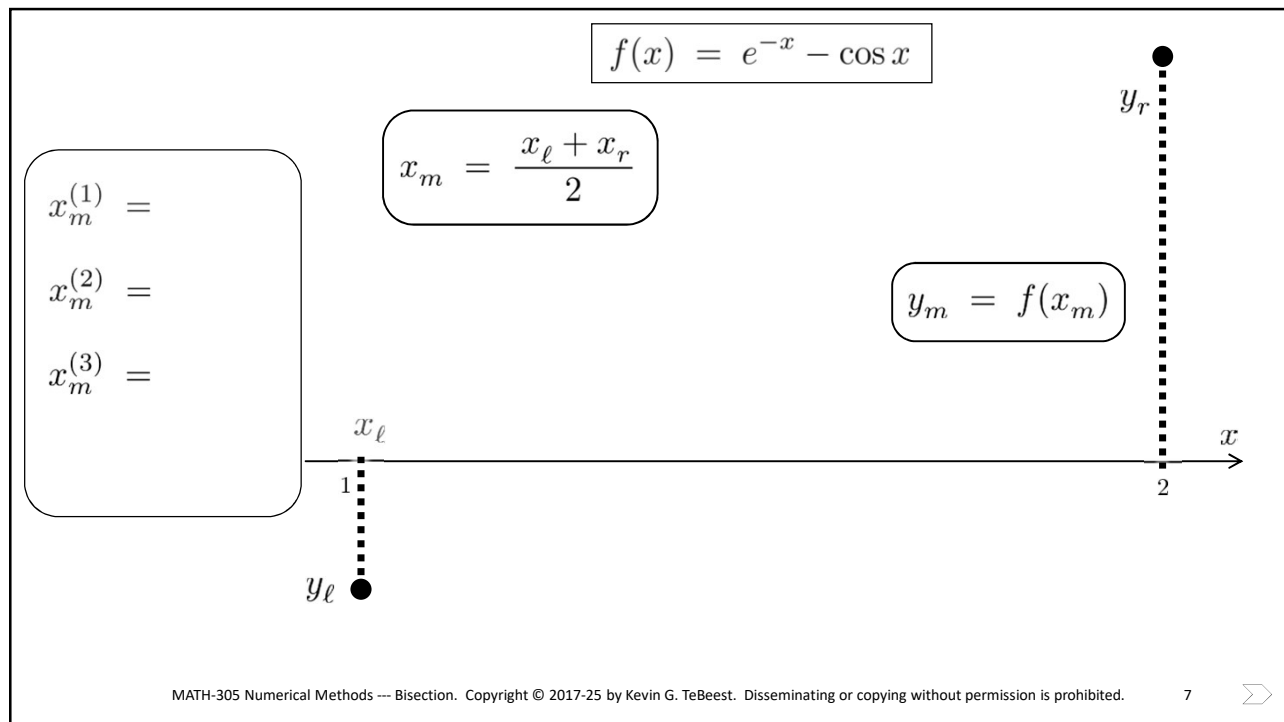
We'll use a starting interval of



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Solve $f(x) = 0$ for x

General Definitions

1. Each x_m , called an _____, approxs the zero z of ftn f .
2. The process of repeating steps to obtain each iterate is called an _____.
3. Performing iterations is called _____.

With enough iterations, the iterates converge to z :

Q . What is

When convenient, we'll let $x_m^{(n)}$ denote the _____ iterate.

E.g., $x_m^{(4)}$ is the _____ iterate.

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Do Ex 2. $f(x) = e^{-x} - \cos x$, $x_m = \frac{x_\ell + x_r}{2}$ Use radians!

Iter $n = 1$:

I always use more digits than I display.

$$\left. \begin{array}{l} x_\ell = 1.0, \quad y_\ell = f(x_\ell) = f(1) = -0.1724\dots, \\ x_r = 2.0, \quad y_r = f(x_r) = f(2) = +0.5514\dots, \end{array} \right\} \text{From the previous slide.}$$

$$x_m = \frac{x_\ell + x_r}{2} = \quad, \quad y_m = f(x_m) = +0.1523\dots$$

$$\text{Sign Test: } \frac{y_\ell \quad y_m \quad y_r}{\quad}$$

So the zero of f is b/w

I.e., the zero of f is on interval



$$x_m = \frac{x_\ell + x_r}{2} \quad f(x) = e^{-x} - \cos x \quad \text{Use radians!}$$

Iter $n = 2$:

$$\left. \begin{array}{l} x_\ell = 1.0, \quad y_\ell = f(x_\ell) = f(1) = -0.1724\dots, \\ x_r = 1.5, \quad y_r = f(x_r) = f(1.5) = +0.1523\dots, \end{array} \right\} \text{From the previous slide.}$$

$$x_m = \frac{x_\ell + x_r}{2} = \quad, \quad y_m = f(x_m) = -0.02881\dots$$

$$\text{Sign Test: } \frac{y_\ell \quad y_m \quad y_r}{\quad}$$

So the zero of f is b/w

I.e., the zero of f is on interval



$$x_m = \frac{x_\ell + x_r}{2} \quad f(x) = e^{-x} - \cos x \quad \text{Use radians!}$$

Iter $n = 3$:

$$\left. \begin{array}{l} x_\ell = 1.25, \quad y_\ell = f(x_\ell) = f(1.25) = -0.02881\dots, \\ x_r = 1.5, \quad y_r = f(x_r) = f(1.5) = +0.1523\dots, \end{array} \right\} \text{From the previous slide.}$$

$$x_m = \frac{x_\ell + x_r}{2} = \quad, \quad y_m = f(x_m) = +0.05829\dots$$

$$\text{Sign Test:} \quad \frac{y_\ell \quad y_m \quad y_r}{\quad}$$

So the zero of f is b/w

I.e., the zero of f is on interval



Use radians!

$$x_m = \frac{x_\ell + x_r}{2} \quad f(x) = e^{-x} - \cos x$$

$$y_\ell = f(1) = -0.1724\ 2286\dots$$

$$y_r = f(2) = +0.5514\ 8211\dots$$

n	x_ℓ	x_m	x_r	y_ℓ	y_m	y_r	$ x_r - x_\ell $
1	1.0000 0000	1.5000 0000	2.0000 0000	—	+0.152392 R	+	0.5000 0000
2	1.0000 0000	1.2500 0000	1.5000 0000	—	-0.028817 L	+	0.2500 0000
3	1.2500 0000	1.3750 0000	1.5000 0000	—	+0.058291 R	+	0.1250 0000
4	1.2500 0000	1.3125 0000	1.3750 0000	—	+0.013712 R	+	0.0625 0000
5	1.2500 0000	1.2812 5000	1.3125 0000	—	-0.007827 L	+	0.0312 5000
6	1.2812 5000	1.2968 7500	1.3125 0000	—	+0.002876 R	+	0.0156 2500

Solve $f(x) = 0$ for x

When do we stop iterating? **As we iterate**, we should see that

- a) the _____ approach _____:
- b) the difference b/w _____ approaches _____:
- c) the _____ of the interval containing zero z decreases to _____:

That means that each of these 3 quantities becomes _____.

Therefore...

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Possible Stopping Tests

Solve $f(x) = 0$ for x

Let TOL, called a _____, denote some _____ positive no. of our choosing. We stop iterating and accept the last x_m as our approx. of zero z when:

- a)
- b)
- c)

IST

The latter is called the _____ (our favorite).

Remember the importance of understanding _____!

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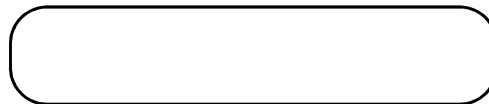
Solve $f(x) = 0$ for x **Observations:**

1. At each iter'n, the _____ of interval $[x_\ell, x_r]$ is _____.
2. Since both x_m and z are on that interval, their difference _____
_____ the interval's length:
3. Consequently, the interval length is _____:

In our ex, the interval length after 6 iter's is _____, so
 (Actual error: $|\text{Error}_6| = 0.004179\dots$)

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Solve $f(x) = 0$ for x **Error Bound** in using x_m to approx zero z :

(EB.1)

So if we use the **interval stopping test** — *stop iterating when*

IST

then our tolerance TOL is an _____:

(EB.2)

So unlike the other stopping tests, with the IST our tolerance TOL _____
 represents the _____ we will _____!

Furthermore:

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Let $L = b - a$ denote the length of the starting interval $[a, b]$.

Then the error:

1. after 1 bisection is

$$|\text{Error}_1| \leq$$

2. after 2 bisections is

$$|\text{Error}_2| \leq$$

3. after 3 bisections is

$$|\text{Error}_3| \leq$$

- n. after n bisections is

$$|\text{Error}_n| \leq$$



Solve $f(x) = 0$ for x

So the error bound after n bisections is

$$|\text{Error}_n| \leq$$

(EB.3)

I.e., the n th iterate $x_m^{(n)}$ must be within _____ units of the zero z of f .



Solve $f(x) = 0$ for x Solve $f(x) = 0$ for x

$$|\text{Error}_n| \leq \frac{b-a}{2^n} \quad (\text{EB.3})$$

Example 3.

Supp. ftn $f(x)$ satisfies the IVT on interval $[-2, 6]$. Determine the error bound if we bisection _____ times.

$$|\text{Error}_7| \leq$$

So the 7th iterate $x_m^{(7)}$ must be within _____ units of zero z .

If we can tolerate that much error, then we bisection 7 times and accept $x_m^{(7)}$.

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**Example 4.**

$$|\text{Error}_n| \leq \frac{b-a}{2^n}$$

Supp. ftn $f(x)$ satisfies the IVT on interval $[-2, 6]$. How many times should we bisection to ensure the error _____ 10^{-3} ?

ANS. We set the error bound to _____ and _____ for _____.

$$\frac{b-a}{2^n} \stackrel{\text{set}}{=} 10^{-3} \implies$$



**Error
Bound**

 $\implies$ $\implies$

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Solve $f(x) = 0$ for x $\Rightarrow$ $\Rightarrow$ $\Rightarrow n =$

So bisecting ____ times ensures that the error will not exceed 10^{-3} units.

Standing
Assignment

**Always rework ALL of my exs
successfully yourself *before*
attempting the HW!**

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Solve $f(x) = 0$ for x

Mathematical Aside:

Real numbers u and v agree to _____ d decimal places if



So if we choose TOL of the form $\text{TOL} = 0.5 \times 10^{-d}$ and stop iterating when

then x_m agrees with z to _____ decimal places.

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Example 5.Solve $f(x) = 0$ for x

Using the starting interval $[-2, 6]$, how many times must we bisect to guarantee _____ decimal place agreement b/w x_m and z ?

$$\begin{aligned}
 |x_r - x_\ell| &\stackrel{\text{set}}{=} \\
 \Rightarrow \frac{b-a}{2^n} &= \\
 \Rightarrow \frac{6 - (-2)}{2^n} &= \\
 \Rightarrow 2^n &= \frac{8}{0.5 \times 10^{-4}} \\
 \Rightarrow &
 \end{aligned}$$

So bisecting _____ **times** ensures that x_m agrees with z to _____ dec. places.

You should rework this ex!

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**Properties of Bisection Mthd**Solve $f(x) = 0$ for x

1. It is an _____ **mthd**. At each iter'n, we possess an interval $[x_\ell, x_r]$ that contains zero z .
2. It gives _____-sided convergence toward zero z :
3. B/c of Prop 2, the _____ shrinks to _____, allowing us to use the IST: Stop iterating when

IST

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Solve $f(x) = 0$ for x

4. B/c of the IST, our tolerance TOL truly represents an error _____:
(EB.2)
5. B/c of the IST, choosing TOL of the form 0.5×10^{-d} ensures that x_m agrees with zero z to _____ dec. places.
6. Converges _____ than other mthds as we'll see.
7. Gives a linear rate of convergence to z :
Digits $\approx kn$.

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Solve $f(x) = 0$ for x

Dec. Place Conv.	Iterations
1	
2	
3	
4	
5	
6	

This mthd gives a _____ rate of convergence:
accuracy is approx. proportional to $n \implies$

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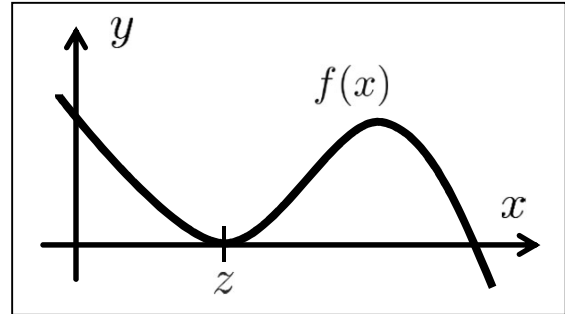
READ. Try to answer these questions *before* reading the answers!



1. Explain why Bisection mthd cannot be used to approx. the zero shown, where ftn f touches the x axis tangentially at zero z without crossing through the x axis.
2. How would you approach this problem?

Answers:

1. If one selects a small starting interval $[a, b]$ containing zero z , then $f(a)$ and $f(b)$ will **not** have opposite signs, so the conditions of the Intermediate Value Thm will not be satisfied.
2. Since f touches the x axis tangentially at $x = z$, then $x = z$ is a zero of both $f(x)$ **and** $f'(x)$. Furthermore, $f'(x)$ changes sign across $x = z$. So we may apply Bisection to ftn $f'(x)$.



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NOTE:



It might be tempting to select our tolerance TOL to be *very* small, such as 10^{-15} , for example. This can be problematic! Recall the number **machine epsilon** from previous notes. It is possible that we select TOL to be **so small** that the the stopping test

$$|x_r - x_\ell| \leq \text{TOL}$$

is **never** satisfied, in which case the code might never stop iterating, even though our iterates $x_m^{(n)}$ become *extremely* close to zero z of $f(x)$.

Recall that you are to **study all** pages of the notes, including those pages not covered in class. They are usually marked with .

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Bisection Method Algorithm

With (x_ℓ, y_ℓ) and (x_r, y_r) in hand:

1. Compute iterate x_m .
2. Compute $y_m = f(x_m)$.
3. Apply the Sign Test (IVT):
 - If y_ℓ and y_m have opp. signs,
 store x_m in x_r and y_m and y_r .
 - Otherwise
 store x_m in x_ℓ and y_m and y_ℓ .
4. Print: iteration number, x_m , and $f(x_m)$.

Repeat Steps 1 – 4 until the IST is satisfied.



Recall that an algorithm is **NOT** a code.

